A BRIEF TOUR OF FERMAT’S CALCULUS

JOHN D. LORCH

Abstract. We examine some of the beautiful and surprisingly substantial contributions which Pierre de Fermat made to the development of calculus. In the process, we emphasize the importance of the limit concept and we introduce alternative viewpoints of both differentiation and integration.

1. Introduction

If Isaac Newton “stood on the shoulders of giants” then, when it came to the development of calculus, he certainly stood on the shoulders of French mathematician Pierre de Fermat (1601-1665). Fermat, a key figure in an era of unprecedented mathematical development, is popularly known for his contributions to number theory and probability. However, he also made beautiful and substantial contributions to the beginnings of calculus.

Our purpose is to present some of Fermat’s work in calculus (cloaked in modern terms) featuring his method of adequality. Fermat’s viewpoint is rich in important ideas, including:

• An alternate approach to differentiation and integration which is both clever and beautiful.
• The importance of a formal limit concept and the Fundamental Theorem of Calculus.
• Yet another illustration of the power of geometric series.
• A foray into the history of mathematics, featuring one of the greatest mathematicians of all time.

2. Ad-equality and Fermat’s tangent line problem

Fermat’s work on the tangent line problem likely began sometime in the 1620’s as an extension of Viete’s work on the theory of equations. Responding to inquiries, in 1636 he prepared a short document outlining his method (see [1]). In the document he describes the following situation: Consider a parabola with corresponding axis of symmetry (see Figure 1). The goal is to draw accurately the tangent to the curve at the point $C$.

The tangent line passes through a point $F$ on the axis of symmetry. Since the tangent is determined by the location of $F$, it suffices to compute the distance $|AF|$, called the subtangent.

To find $|AF|$, begin by observing that $|BD|^2/|AC|^2 = |BF|^2/|AF|^2$ via similar triangles. If $B$ is “close enough” to $A$, then the point $D$ essentially lies on the parabola, and therefore $|BD|^2/|AC|^2$ is essentially equal to $|BE|/|AE|$.

We need to think about what “close enough” means. Fermat handled this situation by setting $e = |AB|$, and then regarding $e$ as a number with special arithmetic properties. Specifically we are to think of $e$ as a small positive number insofar as to
allow all reasonable multiplicative cancellations of \( e \)'s in whatever equations they might appear. Then, after these cancellations have taken place, one is to regard \( e \) as zero\(^1\). This arithmetic with \( e \) has the aura of suspicion, but we proceed as Fermat did.

With \( e = |AB| \) as above, Fermat puts \( |BF|^2/|AF|^2 \approx |BE|/|AE| \), where the symbol \( \approx \) denotes “ad-equality”, meaning to approximate as closely as possible\(^2\). Then, using the arithmetic of \( e \)'s, we compute

\[
\frac{|BF|^2}{|AF|^2} \approx \frac{|BE|}{|AE|} \implies \frac{(|AF| + e)^2}{|AF|^2} \approx \frac{|AE| + e}{|AE|}
\]

\[
\implies 2|AE||AF| + e^2|AE| \approx |AF|^2e
\]

\[
\implies 2|AE||AF| + e|AE| \approx |AF|^2
\]

\[
\implies 2|AE||AF| \approx |AF|^2.
\]

We correctly conclude that the subtangent \( |AF| \) is \( 2|AE| \).

Observe that Fermat’s method is quite general. Consider the the graph of a function \( f(x) \) (see Figure 2). Fermat’s method indicates that the subtangent \( t \) corresponding to the point \((x_0, f(x_0))\) is given by solving \( f(x_0 + e)/f(x_0) \approx (t + e)/t \). With a little more work, we arrive at the suspiciously familiar formula \( (f(x_0 + e) - f(x_0))/e \) for the slope of the tangent line at \((x_0, f(x_0))\).

In order to gain solid footing, the whole procedure described above cries out for a precise limit concept. Why didn’t Fermat simply take a limit as \( e \) approaches zero instead of resorting to a strange and suspect infinitesimal arithmetic? The fact is that Fermat and his contemporaries never arrived at a precise notion of the limit concept. This would have to wait over 200 years for Karl Weierstrass in the late 19-th century, and until then the procedures of calculus, including the one

\(^1\)It was likely never Fermat’s intent that \( e \) be regarded as a mysterious number with special properties as we have described above. However, Fermat’s tract on the tangent method was so short that his motivation for introducing \( e \) was never clearly articulated, and people were left to draw their own conclusions (see [4]).

\(^2\)Fermat stood on the shoulders of giants, too. The term ad-equality comes from the Latin adequatio, which in turn is a translation of the Greek term parisotes used by Diophantus in describing the approximation of a number as closely as possible.
described above, all rested on somewhat shaky theoretical ground (for an account of the development of calculus, see [3]).

3. Geometric series and Fermat’s area problem

Consider the task of finding the area $A$ of the region bounded by the curves $y = x^2$, $y = 0$, and $x = a$. Based on a 1658 account (see [2]), we see that Fermat would proceed as follows.

Let $e$ be the special number described in Section 2, and put $r = 1 - e$. If for a moment we think of $e$ as a small positive number, then the values $a, ar, ar^2, \ldots$ form an infinite partition of the interval $[0, a]$. Above each subinterval $[ar^n, ar^{n+1}]$, we may construct an approximating rectangle $R_n$, the height of which is $(ar^n)^2$ (see Figure 3)). Observe that if $e$ is very small, then $R_n$ closely approximates the corresponding region under the curve.
Let $A_n$ denote the area of $R_n$, and note $A_n = r^{3n}A_0$. Then via ad-equality we have

$$A \approx \sum_{n=0}^{\infty} A_0(r^3)^n,$$

which Fermat recognizes as a garden-variety geometric series with first term is $A_0 = a^3(1 - r)$ and common ratio $r^3$. Summing the series, recalling that $r = 1 - e$, and performing adequality arithmetic, we obtain

$$A \approx \frac{a^3(1 - r)}{1 - r^3} \approx \frac{ea^3}{1 - (1 - e)^3} \approx \frac{ea^3}{3e - 3e^2 + e^3} \approx \frac{a^3}{3 - 3e + e^2} \approx \frac{a^3}{3}.$$

We correctly conclude that $A = a^3/3$.

Fermat observed that this beautiful argument generalizes, and was able to correctly compute $\int_0^a x^r \, dx$, where $r$ is any positive rational number\(^3\).

4. Did Fermat invent calculus?

Our discussion begs the question: Did Fermat invent calculus? At first glance, the answer may be “yes”. After all, in calculus we compute tangent lines and areas. However, there is something missing here. Apparently Fermat never considered the inverse problem associated to the tangent line problem. That is, given information about the tangents to a curve, can one completely describe the curve itself? It was up to Newton (and others) to realize that the solution to this inverse problem is encoded in a seemingly unlikely place, namely the solution to the area problem! This eventually gave rise to the Fundamental Theorem of Calculus, and it is for this great achievement that we give Newton and Leibniz the credit for “inventing” calculus.

References


Department of Mathematics, Ball State University, Muncie, IN 47306
E-mail address: jdlorch@math.bsu.edu

\(^3\)This generalizes Bonaventura Cavalieri’s computation of $\int_0^a x^n \, dx$ for whole number values of $n$